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# On Strong and Ordinary Summability

by

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## §1. Introduction

Let $\{p_n\}$ be a sequence of non-negative numbers with $p_0>0$, $P_n:=p_0+\cdots+p_n\to\infty$, and for a given sequence $s=\{s_n\}$ let

$$(M_p s)_n:=P_n^{-1}\sum_{k=0}^{n} p_k s_k \qquad \text{for} \quad n=0,1,\cdots.$$

We say that $s$ is $M_p$-limitable to $\sigma$ ($M_p$-$\lim s_n=\sigma$), if $(M_p s)_n\to\sigma$, and we say that $s$ is strongly $M_p$-limitable with index $\lambda>0$ to $\sigma$ ($[M_p]_\lambda$-$\lim s_n=\sigma$), if $M_p$-$\lim|s_n-\sigma|^\lambda=0$.

The methods $M_p$ and $[M_p]_\lambda$ are regular, i.e. we have $M_p$-$\lim s_n=\sigma$ and $[M_p]_\lambda$-$\lim s_n=\sigma$, whenever $\lim s_n=\sigma$ holds, and, if $\lambda\geq 1$ then

$$[M_p]_\lambda\text{-}\lim s_n=\sigma \Longrightarrow M_p\text{-}\lim s_n=\sigma \tag{1.1}$$

(see for example Borwein [2; Theorem 3 and its corollary (I)]). Thus the following result of Goto and Kano [3; Lemma 1] can be considered as a Tauberian theorem of $M_p\to[M_p]_1$-type.

THEOREM GK. *Let $M_p$-$\lim s_n=\sigma$ and*

$$\sum_{k=1}^{n} P_{k-1}|a_k|=O(P_n)\,. \tag{1.2}$$

*Then $[M_p]_1$-$\lim s_n=\sigma$.*

Here and in the following let $a_n=s_n-s_{n-1}$ ($s_{-1}:=0$), and the Landau symbol $O$ has its usual meaning. Theorem GK generalizes a result of Karamata [5; p. 123], who proved it for $M_p=C_1$, the Cesàro method of order 1. The present paper was suggested by Theorem GK and its proof. In §3 we show for example that in Theorem GK the assertion may be replaced by $[M_p]_\lambda$-$\lim s_n=\sigma$ for each $\lambda>0$ (Theorem 3.4), and §4 for example contains a Tauberian theorem of $J_p\to[M_p]_\lambda$-type (Theorem 4.3), where $J_p$ denotes a general power series method.

## §2. Notations

All sequences $\{x_n\}$ are defined for $n=0,1,\cdots$ if nothing else is said, and $x_{-1}$

is defined as 0. In any case, with the exception of the proof of Theorem 3.4., $s=\{s_n\}$ denotes a sequence of complex numbers, and for a given sequence $s$ let $s^r:=\{s_n^r\}$ for each $r\in N_0:=\{0, 1, 2, \cdots\}$, especially $0^0:=1$. We use the usual symbols $m:=\{s\colon \sup|s_n|<\infty\}$ and $c_0:=\{s\colon \lim s_n=0\}$. The letter $M_p$ is also used for the matrix which defines the method $M_p$ or $[M_p]_\lambda$. For $r\in N:=\{1, 2, \cdots\}$ let $M_p^r$ be the $r$-th power of the matrix $M_p$, and if $p_n>0$, let $M_p^{-1}$ be the inverse of $M_p$. We occasionally write $M_p^0$ instead of $I$, the identity matrix.

## §3. Weighted mean methods

For our generalization of Theorem GK we need some Lemmas.

LEMMA 3.1. *Let $r\in N_0$ and $n\in N_0$. Then*

$$P_n^{-1}\sum_{k=1}^{n}(M_ps^r)_{k-1}P_{k-1}a_k=(M_ps^r)_ns_n-(M_ps^{r+1})_n\,. \tag{3.1}$$

*Proof.* For an arbitrary sequence $\{x_n\}$ we have

$$P_n^{-1}\sum_{k=1}^{n}x_{k-1}a_k=P_n^{-1}\sum_{k=1}^{n}x_{k-1}(s_k-s_{k-1})=P_n^{-1}\{x_ns_n-\sum_{k=0}^{n}(x_k-x_{k-1})s_k\}\,.$$

From this, with $x_n:=(M_ps^r)_nP_n$, we obtain (3.1).

For $r=0$ equation (3.1) is the well-known Kronecker formula

$$P_n^{-1}\sum_{k=1}^{n}P_{k-1}a_k=s_n-(M_ps)_n=:\delta_n\,. \tag{3.2}$$

LEMMA 3.2. *Let $r\in N_0$ and $n\in N_0$. Then*

$$(M_ps^{r+1})_n=(M_ps^r)_n(M_ps)_n+P_n^{-1}\sum_{k=1}^{n}\{(M_ps^r)_n-(M_ps^r)_{k-1}\}P_{k-1}a_k\,.$$

*Poof.* From (3.1) we obtain

$$(M_ps^{r+1})_n=(M_ps^r)_ns_n-P_n^{-1}\sum_{k=1}^{n}(M_ps^r)_{k-1}P_{k-1}a_k\,.$$

Thus we have to show

$$P_n^{-1}(M_ps^r)_n\sum_{k=1}^{n}P_{k-1}a_k=(M_ps^r)_ns_n-(M_ps^r)_n(M_ps)_n\,.$$

But this is trivial, if $(M_ps^r)_n=0$, and is equivalent to (3.2), if $(M_ps^r)\neq 0$.

THEOREM 3.3. *Let $M_p$-$\lim s_n=\sigma$ and* (1.2). *Then*

$$M_p\text{-}\lim s_n^r=\sigma^r \qquad \textit{for each} \quad r\in N_0\,. \tag{3.3}$$

*Proof.* The three cases $r=0$, $r=1$ and $a_n=0$ for $n>0$ are trivial. So let $a_n\neq 0$

for at least one $n>0$ and suppose (3.3) to hold for a fixed $r>0$. Then

$$(3.4) \qquad (M_p s^r)_n (M_p s)_n \to \sigma^{r+1}.$$

For a given $\varepsilon>0$ choose $\kappa$ such that

$$|(M_p s^r)_n - (M_p s^r)_{k-1}| < \varepsilon / \sup P_n^{-1} \sum_{k=1}^{n} P_{k-1} |a_k| \qquad \text{for} \quad n \geq k \geq \kappa.$$

Then, for $n \geq k \geq \kappa$

$$|P_n^{-1} \sum_{k=1}^{n} \{(M_p s^r)_n - (M_p s^r)_{k-1}\} P_{k-1} a_k|$$
$$\leq P_n^{-1} \sum_{k=1}^{k-1} |(M_p s^r)_n - (M_p s^r)_{k-1}| P_{k-1} |a_k| + \varepsilon.$$

Thus

$$(3.5) \qquad P_n^{-1} \sum_{k=1}^{n} \{(M_p s^r)_n - (M_p s^r)_{k-1}\} P_{k-1} a_k \to 0$$

by $P_n \to \infty$, and we obtain $M_p$-$\lim s_n^{r+1} = \sigma^{r+1}$ by (3.4), (3.5) and Lemma 3.2.

The case $M_p = C_1$, $r=2$ of Theorem 3.3 is due to Karamata [5; page 123], whereas for $r=2$ we obtain a part of the proof of Theorem GK [3; page 86].

THEOREM 3.4. *Let $M_p$-$\lim s_n = \sigma$ and* (1.2). *Then $[M_p]_\lambda$-$\lim s_n = \sigma$ for each $\lambda > 0$.*

*Proof.* If $\lambda > \mu > 0$, then $[M_p]_\lambda$-$\lim s_n = \sigma \Rightarrow [M_p]_\mu$-$\lim s_n = \sigma$ by Borwein [2; Theorem 1]. Therefore it is enough to prove Theorem 3.4 for the case that $\lambda$ is an even positive integer. We further may assume without loss of generality that the $s_n$ are real. Then we obtain

$$P_n^{-1} \sum_{k=0}^{n} p_k |s_k - \sigma|^\lambda = P_n^{-1} \sum_{k=0}^{n} p_k (s_k - \sigma)^\lambda = \sum_{\nu=0}^{\lambda} \binom{\lambda}{\nu} (-1)^{\lambda-\nu} \sigma^{\lambda-\nu} (M_p s^\nu)_n,$$

and thus, by Theorem 3.3,

$$P_n^{-1} \sum_{k=0}^{n} p_k |s_k - \sigma|^\lambda \to \sigma^\lambda \sum_{\nu=0}^{\lambda} (-1)^\nu \binom{\lambda}{\nu} = 0.$$

The case $M_p = C_1$, $r=2$ of Theorem 3.4 is due to Karamata [5; page 123], whereas for $r=2$ we obtain Theorem GK.

The example $M_p := C_1$, $s_n := (-1)^n$ shows that (1.2) in Theorem 3.4 may not even be omitted for $0 < \lambda < 1$.

Our next aim is an extension of Theorem 3.4. Let $Q$ be a matrix. Following Borwein [2] we say that the sequence $s$ is strongly $[M_p, Q]_\lambda$-limitable to $\sigma$ ($[M_p, Q]_\lambda$-$\lim s_n = \sigma$), if $Qs$ exists and $M_p$-$\lim |(Qs)_n - \sigma|^\lambda = 0$. For example, $[M_p, I]_\lambda$-$\lim s_n = \sigma$ is identical to $[M_p]_\lambda$-$\lim s_n = \sigma$.

Using $\delta=\{\delta_n\}$ from (3.2), we obtain

THEOREM 3.5. *Let* $r\in N$, $M_p^{r+1}$-$\lim s_n=\sigma$ *and*

$$\sum_{k=1}^{n} p_k|(M_p^{r-1}\delta)_k|=O(P_n)\,. \tag{3.6}$$

*Then* $[M_p, M_p^r]_\lambda$-$\lim s_n=\sigma$ *for each* $\lambda>0$.

*Proof.* If we replace $s$ in Theorem 3.4 by $M_p^r s$, then $a_k$ changes into

$$\begin{aligned}(M_p^r s)_k-(M_p^r s)_{k-1}&=P_k^{-1}\sum_{\nu=0}^{k} p_\nu(M_p^{r-1}s)_\nu-P_{k-1}^{-1}\sum_{\nu=0}^{k-1} p_\nu(M_p^{r-1}s)_\nu\\ &=P_k^{-1}p_k(M_p^{r-1}s)_k+(P_k^{-1}-P_{k-1}^{-1})\sum_{\nu=0}^{k-1} p_\nu(M_p^{r-1}s)_\nu\\ &=P_k^{-1}p_k\{(M_p^{r-1}s)_k-(M_p^r s)_{k-1}\}\\ &=P_k^{-1}p_k\{(M_p^{r-1}s)_k-(M_p^r s)_k\}+P_k^{-1}p_k\{(M_p^r s)_k-(M_p^r s)_{k-1}\}\,,\end{aligned}$$

from which we obtain

$$(M_p^r s)_k-(M_p^r s)_{k-1}=P_{k-1}^{-1}p_k\{(M_p^{r-1}s)_k-(M_p^r s)_k\}=P_{k-1}^{-1}p_k(M_p^{r-1}\delta)_k\,.$$

Thus (1.2) changes into (3.6), and Theorem 3.5 follows from Theorem 3.4.

If $p_n>0$ for each $n$, then in Theorem 3.5 also $r=0$ is admissible and results in Theorem 3.4 because $(M_p^{-1}\delta)_n=P_{n-1}p_n^{-1}a_n$.

THEOREM 3.6. *Let* $r\in N_0$, $\lambda>0$, $[M_p, M_p^{r+1}]_\lambda$-$\lim s_n=\sigma$ *and*

$$[M_p, M_p^r]_\lambda\text{-}\lim \delta_n=0\,. \tag{3.7}$$

*Then* $[M_p, M_p^r]_\lambda$-$\lim s_n=\sigma$.

*Proof.* Let $Q$ be a matrix. Then from Minkowski's inequality in the case $\lambda\geq 1$ and the corresponding inequality in the case $0<\lambda<1$ we obtain

$$[M_p, Q]_\lambda\text{-}\lim s_n=\sigma\wedge[M_p, I-Q]_\lambda\text{-}\lim s_n=0\Longrightarrow[M_p]_\lambda\text{-}\lim s_n=\sigma\,.$$

From this, replacing $Q$ and $s$ by $M_p$ and $M_p^r s$ respectively, Theorem 3.6 follows.

Because of the regularity of $M_p$ we may replace $[M_p, M_p^{r+1}]_\lambda$-$\lim s_n=\sigma$ in Theorem 3.6 by the stronger assumption $M_p^{r+1}$-$\lim s_n=\sigma$ (see Borwein [2; Theorem 3(i)]). Now, comparing Theorem 3.6 with Theorems 3.4 and 3.5, the question arises, whether the conditions $M_p$-$\lim|\delta_n|^\lambda=0$ and (1.2) (in the case $r=0$) and the conditions (3.7) and (3.6) (in the case $r\in N$) are comparable. In general they are not, as we can see for $M_p=C_1$:

a) Let $a_n:=(-1)^n n^{-1/2}$ for $n\in N$. Then $\delta_n\to 0$, and $[C_1]_\lambda$-$\lim\delta_n=0$ for each $\lambda>0$, but (1.2) does not hold.

b) Let $a_n:=n^{-1}$ for $n\in N$. Then (1.2) is fulfilled, but we have $[C_1]_\lambda$-$\lim\delta_n=1$ for each $\lambda>0$.

c) Choose $a_n$ such that $\delta_n = 1$ for $n > 0$. Then $C_1^{r-1}\delta \in m$ and hence $C_1\{|(C_1^{r-1}\delta)_n|\} \in m$ for each $r \in \boldsymbol{N}$. But $|(C_1^r\delta)_n|^\lambda = 1$ for each $n \in \boldsymbol{N}$ and each $\lambda > 0$ and thus $[C_1, C_1^r]_\lambda$-lim $\delta_n = 0$ does not hold.

d) Choose $a_n$ such that $C_1^{r-1}\delta = \{(-1)^n n^{1/2}\}$. Then $|(C_1^{r-1}\delta)_n| \to \infty$ and hence $C_1\{|(C_1^{r-1}\delta)_n|\} \notin m$. But $C_1^r\delta \in c_0$ and thus $[C_1, C_1^r]_\lambda$-lim $\delta_n = 0$.

If $\lambda \geq 1$, Theorem 3.6 is reversible.

Theorem 3.7. *Let* $r \in \boldsymbol{N}_0$ *and* $\lambda \geq 1$. *Then*

$$[M_p, M_p^{r+1}]_\lambda\text{-lim } s_n = \sigma \wedge [M_p, M_p^r]_\lambda\text{-lim } \delta_n = 0 \Longleftrightarrow [M_p, M_p^r]_\lambda\text{-lim } s_n = \sigma .$$

*Proof.* The implication from left to right is a part of Theorem 3.6. Now let $[M_p, M_p^r]_\lambda$-lim $s_n = \sigma$, i.e. $[M_p]_\lambda$-lim $(M_p^r s)_n = \sigma$. Then $M_p$-lim $(M_p^r s)_n = \sigma$ by (1.1), i.e. $M_p^{r+1}$-lim $s_n = \sigma$, and thus $[M_p, M_p^{r+1}]_\lambda$-lim $s_n = \sigma$ by the regularity of $M_p$. Hence we have $M_p$-lim $|(M_p^r s)_n - \sigma|^\lambda = 0$ and $M_p$-lim $|(M_p^{r+1}s)_n - \sigma|^\lambda = 0$, and therefore

$$[M_p, M_p^r]_\lambda\text{-lim } \delta_n = M_p\text{-lim } |(M_p^r s)_n - (M_p^{r+1}s)_n|^\lambda = 0 ,$$

because, by a well-known inequality,

$$|(M_p^r s)_n - (M_p^{r+1}s)_n|^\lambda \leq 2^{\lambda-1}(|(M_p^r s)_n - \sigma|^\lambda + |(M_p^{r+1}s)_n - \sigma|^\lambda) .$$

Using the regularity of $M_p$ and a part of the above proof we obtain

Corollary 3.8. *Let* $r \in \boldsymbol{N}_0$ *and* $\lambda \geq 1$. *Then* $M_p^{r+1}$*-lim* $s_n = \sigma \wedge [M_p, M_p^r]_\lambda$*-lim* $\delta_n = 0 \Leftrightarrow [M_p, M_p^r]_\lambda$*-lim* $s_n = \sigma$.

## §4. Power series methods

With $\{p_n\}$ as defined in §1 we say that $\{s_n\}$ is $J_p$-limitable to $\sigma$ ($J_p$-lim $s_n = \sigma$), if, with $x \in \boldsymbol{R}$, the series

$$p(x) := \sum_{n=0}^{\infty} p_n x^n$$

has radius of convergence 1, the series

$$p_s(x) := \sum_{n=0}^{\infty} p_n s_n x^n$$

converges for $0 < x < 1$ and $\lim_{x \to 1-} p_s(x)/p(x) = \sigma$. In the following, we need the additional condition

$$P_n/P_m \to 1 \quad \text{as} \quad n/m \to 1 \qquad (n > m \to \infty) \tag{4.1}$$

for the sequence $\{P_n\}$. Then we have

Lemma 4.1 ([7; Satz 3.7]). *Let* (4.1), $J_p$-lim$s_n = \sigma$ *and*

(4.2) $$\limsup |s_n - s_m| < \infty \quad as \quad P_n/P_m \to 1 \qquad (n > m \to \infty)\,.$$

*Then* $M_p$-$\lim s_n = \sigma$.

THEOREM 4.2. *Let* (4.1), $J_p$-$\lim s_n = \sigma$ *and* (1.2). *Then* $M_p$-$\lim s_n = \sigma$.

*Proof.* Using Lemma 4.1 it is sufficient to show (1.2)⇒(4.2): By (1.2), the sequence $b = \{b_n\}$ with

$$b_n := P_n^{-1} \sum_{k=1}^{n} P_{k-1} |a_k|$$

is bounded, we obtain $|a_n| = (b_n - b_{n-1}) + P_{n-1}^{-1} p_n b_n$, and thus, for $n > m$,

$$\begin{aligned} |s_n - s_m| &\le \sum_{k=m+1}^{n} |a_k| = (b_n - b_m) + \sum_{k=m+1}^{n} P_{k-1}^{-1} p_k b_k \\ &\le O(1) + P_n P_m^{-1} (M_p b)_n = O(1) \quad \text{as} \quad P_n/P_m \to 1\,. \end{aligned}$$

From this (4.2) follows.

From Theorems 4.2 and 3.4 we obtain

THEOREM 4.3. *Let* (4.1), $J_p$-$\lim s_n = \sigma$ *and* (1.2). *Then* $[M_p]_\lambda$-$\lim s_n = \sigma$ *for each* $\lambda > 0$.

If, for $r \in N$, we write $J_p M_p^r$-$\lim s_n = \sigma$ instead of $J_p$-$\lim (M_p^r s)_n = \sigma$, we have

THEOREM 4.4. *Let* $r \in N$, (4.1), $J_p M_p^r$-$\lim s_n = \sigma$ *and* (3.6). *Then* $[M_p, M_p^r]_\lambda$-$\lim s_n = \sigma$ *for each* $\lambda > 0$.

*Proof.* In the proof of Theorem 3.5 we can see that (1.2) changes into (3.6), if we replace $s$ by $M_p^r s$. Thus we obtain $M_p^{r+1}$-$\lim s_n = \sigma$ from Theorem 4.2 (replacing $s$ by $M_p^r s$) and then $[M_p, M_p^r]_\lambda$-$\lim s_n = \sigma$ for each $\lambda > 0$ by Theorem 3.5.

If $p_n > 0$ for each $n$, then in Theorem 4.4 also $r = 0$ is admissible and results in Theorem 4.3 because $(M_p^{-1} \delta)_n = P_{n-1} p_n^{-1} a_n$.

For many sequences $\{p_n\}$ we have

(4.3) $$J_p\text{-}\lim s_n = \sigma \Longrightarrow J_p M_p\text{-}\lim s_n = \sigma\,.$$

In these cases we may replace $J_p M_p^r$-$\lim s_n = \sigma$ in Theorem 4.4 by the assumption $J_p$-$\lim s_n = \sigma$.

Condition (4.3) for example holds for $p_n := \binom{n+\alpha}{n}$ ($\alpha > -1$ fixed) (see Amir (Jakimovski) [1; Theorem (8.3)]) because in this case $J_p$ is the generalized Abel method $A_\alpha$ ($A_0$ is the classical Abel method) and the corresponding method $M_p$, which we denote by $M_{\alpha+1}$ is a Hausdorff method (see Hausdorff [4; page 88]). We further have $P_n=\binom{n+\alpha+1}{n}$, and (4.1) holds. Therefore, if in this case we write $\delta_{\alpha+1}$ instead of $\delta$, from Theorem 4.3 and 4.4 and the following remark we obtain:

COROLLARY 4.5. *Let* $A_0$-$\lim s_n=\sigma$ *and*

$$\sum_{k=1}^{n} k|a_k|=O(n+1)\,. \tag{4.4}$$

*Then* $[C_1]_\lambda$-$\lim s_n=\sigma$ *for each* $\lambda>0$.

COROLLARY 4.6. *Let* $r\in N$, $A_\alpha$-$\lim s_n=\sigma$ *and*

$$\binom{n+\alpha+1}{n}^{-1}\sum_{k=1}^{n}\binom{k+\alpha}{k}|(M_{\alpha+1}^{r-1}\delta_{\alpha+1})_k|=O(1)\,. \tag{4.5}$$

*Then* $[M_{\alpha+1},M_{\alpha+1}^r]_\lambda$-$\lim s_n=\sigma$ *for each* $\lambda>0$.

Now we have $M_1=C_1$, and each method $M_{\alpha+1}$ is equivalent to $C_1$, i.e. for each $\alpha>-1$

$$M_{\alpha+1}\text{-}\lim s_n=\sigma \Longleftrightarrow C_1\text{-}\lim s_n=\sigma \tag{4.6}$$

holds by Hausdorff [4; page 88] (see also [8; (3.1)]) as well as

$$(M_{\alpha+1}s)_n\in m \Longleftrightarrow (C_1 s)_n\in m \tag{4.7}$$

by Knopp [6] and Winn [9] (see also [8; Korollar 4.2]). Therefore the question arises, whether we may replace $M_{\alpha+1}$ in Corollary 4.6 by $C_1$. The answer is in the affirmative:

COROLLARY 4.7. *Let* $r\in N$, $A_\alpha$-$\lim s_n=\sigma$ *and*

$$\sum_{k=1}^{n}|(C_1^{r-1}\delta_1)_k|=O(n+1)\,. \tag{4.8}$$

*Then* $[C_1,C_1^r]_\lambda$-$\lim s_n=\sigma$ *for each* $\lambda>0$.

*Proof.* By Borwein [2; Theorem 1] it is enough to show

$$[C_1,C_1^r]_\lambda\text{-}\lim s_n=\sigma \qquad \text{for each} \quad \lambda\geq 1\,. \tag{4.9}$$

Thus, using Corollary 4.6, we have to show that (4.8) implies (4.5) and that

$$[M_{\alpha+1},M_{\alpha+1}^r]_\lambda\text{-}\lim s_n=\sigma \qquad \text{for each} \quad \lambda\geq 1 \tag{4.10}$$

implies (4.9). Now (4.8) is equivalent to

$$\binom{n+\alpha+1}{n}^{-1}\sum_{k=1}^{n}\binom{k+\alpha}{k}|(C_1^{r-1}\delta_1)_k|=O(1) \tag{4.11}$$

by (4.7). We further have $M_{\alpha+1}C_1=C_1M_{\alpha+1}$ because Hausdorff matrices commute.

Thus $C_1^{r-1}$ and $M_{\alpha+1}^{r-1}$ are equivalent regular Hausdorff matrices. Hence there exists a regular Hausdorff matrix $H$ with $M_{\alpha+1}^{r-1}=HC_1^{r-1}$. Now following the proof of Borwein [2; Theorem 5] (with $\lambda=1$ and $M_{\alpha+1}$, $C_1^{r-1}$, $H$ instead of $P$, $Q$, $X$ respectively and regarding the fact that from "$\tilde{X}(u_n)\to 0$ whenever $u_n\to 0$" in Borwein's proof follows, that "$\{\tilde{X}(u_n)\}\in m$ whenever $\{u_n\}\in m$" holds) we get

$$\binom{n+\alpha+1}{n}^{-1}\sum_{k=1}^{n}\binom{k+\alpha}{k}|(M_{\alpha+1}^{r-1}\delta_1)_k|=O(1)$$

from (4.11), i.e. there is a constant $K>0$ so that

$$\sup_n\binom{n+\alpha+1}{n}^{-1}\sum_{k=1}^{n}\binom{k+\alpha}{k}|(M_{\alpha+1}^{r-1}\delta_1)_k|\le K. \tag{4.12}$$

But from (4.12) we also obtain

$$\sup_n\binom{n+\alpha+1}{n}^{-1}\sum_{k=1}^{n}\binom{k+\alpha}{k}|(M_{\alpha+1}^{r}\delta_1)_k|\le K, \tag{4.13}$$

because for each $k\in N$ we have

$$|(M_{\alpha+1}^{r}\delta_1)_k|\le\binom{k+\alpha+1}{k}^{-1}\sum_{\nu=1}^{n}\binom{\nu+\alpha}{\nu}|(M_{\alpha+1}^{r-1}\delta_1)_\nu|\le K.$$

We further have the identity

$$C_1-M_{\alpha+1}=-\alpha(\alpha+1)^{-1}M_{\alpha+1}(I-C_1) \tag{4.14}$$

(see [8; (2.3)]). Hence we obtain from (4.12) to (4.14):

$$\begin{aligned}
&\binom{n+\alpha+1}{n}^{-1}\sum_{k=1}^{n}\binom{k+\alpha}{k}|(M_{\alpha+1}^{r-1}\delta_{\alpha+1})_k|\\
&\le\binom{n+\alpha+1}{n}^{-1}\sum_{k=1}^{n}\binom{k+\alpha}{k}|(M_{\alpha+1}^{r-1}\delta_1)_k+[M_{\alpha+1}^{r-1}(C_1-M_{\alpha+1})s]_k|\\
&\le K+|\alpha|(\alpha+1)^{-1}\binom{n+\alpha+1}{n}^{-1}\sum_{k=1}^{n}\binom{k+\alpha}{k}|(M_{\alpha+1}^{r}\delta_1)_k|\\
&\le K(1+|\alpha|(\alpha+1)^{-1}),
\end{aligned}$$

and thus (4.5) holds. Finally from (4.10) and (4.6) we obtain $[C_1, M_{\alpha+1}^r]_\lambda$-$\lim s_n=\sigma$, and from this we get (4.9) by a direct application of Borwein [2; Theorem 5].

A second example for a sequence $\{p_n\}$, which fulfills (4.1) and (4.3) is $p_n:=(n+1)^{-1}$. The corresponding methods $J_p$ and $M_p$ are the logarithmic methods $L$ and $l$ respectively, and, with

$$\alpha_n:=\sum_{k=0}^{n}(k+1)^{-1},$$

from Theorems 4.3, 4.4 and the following remark we obtain:

COROLLARY 4.8. *Let* $L$-$\lim s_n=\sigma$ *and* $\sum_{k=1}^{n} \alpha_{k-1} |a_k| = O(\alpha_n)$. *Then* $[l]_\lambda$-$\lim s_n=\sigma$ *for each* $\lambda>0$.

COROLLARY 4.9. *Let* $r \in N$, $L$-$\lim s_n=\sigma$ *and* $\sum_{k=1}^{n} (k+1)^{-1} |(l^{r-1}\delta)_k| = O(\alpha_n)$. *Then* $[l, l^r]_\lambda$-$\lim s_n=\sigma$ *for each* $\lambda>0$.

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